Detrending Time-Aggregated Data

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Abstract

This paper examines the combined influences of detrending and time aggregation on the measurement of business cycles. The approximate band-pass filter of Baxter and King (1999) performs relatively well in the sense that it retains the basic shape of disaggregate spectra and cospectra when applied to time aggregated data and is straightforward to apply across sampling intervals. Analysis of known time series processes and actual U.S. macro data, as well as simulation of a standard high-frequency RBC model, confirm the theoretical results.

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1 Introduction

One of the most challenging and controversial aspects of business-cycle research is attempting to remove secular trends from macro data, and do so in a manner that allows one to disentangle the cyclical and growth components. Much of the controversy stems from the fact that seemingly innocuous data transformations have been shown to frequently induce undesirable behavior in the residual series. Kuznets' work in the 1960's is a classic example of how well-intended data transformations can lead to spurious results, such as his discovery of 20-year long swings in economic activity (Adelman (1965); Howrey (1968)). A second example comes from Nelson and Kang (1981), who show that the residuals from a regression of a random walk on time display spurious periodicity. The upside of these and other related studies is that practioneers are now more aware of the possible pitfalls of detrending data. The downside is that, in the process of searching for better detrending procedures, the number of alternative detrending methods have grown rapidly. On the surface, it would seem that having more detrending options available would be advantageous, but unfortunately these procedures often produce widely varying pictures of the business-cycle and economic theory is rarely sharp enough to make recommendations about which method is appropriate (Canova (1998)). Therefore, empirical macroeconomists are often left to choose from the various detrending options with little guidance. This paper attempts to shed further light on which detrending methods are most desirable by examining their ability to detrend temporally aggregated data without distorting the underlying behavioral properties of the economic series. This is an important criterion because nearly all macroeconomic data are heavily aggregated over time. The focus is on three of the most commonly used detrending methods in empirical macroeconomics: the first difference (FD), the Hodrick-Prescott (HP), and the Baxter-King (BK) approximate band-pass filters.

Over the last two decades, the FD filter has been a popular method for removing the trend from nonstationary time series. This is due in large part to Nelson and Plosser (1982), who argue that many macroeconomic time series are difference, rather than trend stationary. When a series is measured in natural logarithms, the resulting FD-filtered series are approximate growth rates and have been commonly used to represent the business-cycle fluctuations of a time series. The problem, however, with treating growth rates of series as business-cycle fluctuations is that the FD filter tends to exacerbate high-frequency noise and introduce a phase shift. Despite these criticisms, many prominent studies of business-cycle phenomena continue to examine the growth rates of macro series (e.g., Campbell (1999), Cogley and Nason (1995a), and Plosser (1989)). An advantage of the HP filter, relative to the FD filter, is that it does not exacerbate high-frequency noise and does not introduce a phase shift into the data. As a consequence, the HP filter, introduced by Hodrick and Prescott (1980), has arguably become the "industry standard" for detrending data in empirical macroeconomics. This flexible detrending method trades off deviations from trend against an adjustable smoothness criterion. The HP filter, however, is not without its critics. It has been criticized for generating spurious cycles in difference stationary data (Harvey and Jaeger (1993); Cogley and Nason (1995b)); altering the persistence, variability and co-movement of time series (King and Rebelo (1993)); and (similar to the FD filter) for passing through high-frequency or "irregular" variation. The third detrending method I consider is the BK filter. This procedure, introduced by Baxter and King (1999), is based on the well-established concept of a spectral band-pass filter. A primary advantage of the BK filter is that it is more consistent with the Burns and Mitchell (1946) definition of the business cycle as containing cycles that last between six and 32 quarters. Although relatively new, the BK filter appears to be gaining in popularity and has recently been employed by studies such as Basu and Taylor (1999) and by Stock and Watson (1999) in the *Handbook of Macroeconomics* to document various business-cycle facts.

In addition to filtering out a trend, most macro time series are also implicitly passed through a time-aggregation (TA) filter. For institutional reasons, business-cycle researchers work almost exclusively with quarterly data. Since data-generating processes for economic data are generally thought to operate at continuous or very short discrete-time intervals, observed quarterly data are therefore some sort of transformation of the high-frequency data. If the variable is a flow, then it is typically summed (or averaged) over shorter time intervals. If it is a stock, then it is typically systematically sampled. Ideally, we would like time aggregation to preserve as many of the underlying properties of the high-frequency data-generating process as possible. This and other aspects of time aggregation have been extensively explored both in the time domain (e.g., Tiao (1972); Weiss (1984); and Rossana and Seater (1992)) and in the frequency domain (e.g., Sims

(1971) and Granger and Siklos (1995)).

The unique contribution of this paper is to examine how the *joint* application of detrending and TA filters influence how we look at business cycles. As mentioned above, there are many studies that examine the individual effects of detrending and time aggregation, but despite the fact that detrending and time aggregation are almost always joint phenomena, few have examined their combined effects. The results in this paper suggest that the interaction between the two filters is important for how we measure business cycles and test our models. The criteria used in this paper to contrast the various detrending filters is (1) their ability to maintain the spectral properties of time series generated by economic decisions at shorter, more frequent intervals and (2) their ease of use across frequencies. Recent research (e.g., Heaton (1993), Chari, Kehoe and McGratten (2000), and Aadland (2001)) has shown that temporal aggregation of data generated at higher frequencies can have important effects on the behavior of economic time series and evaluation of various models. The argument is not that we need to employ or collect data at more frequent intervals, as that may introduce unnecessary measurement error, but rather that we need to explicitly model that sampling and aggregation process that is implicit in almost all macro data. This can be done by solving and calibrating macroeconomic models at higher frequencies and aggregating them up to the data-sampling interval. The primary concern in this paper is how does the time aggregation that is implicit in the actual data and explicit in the artificial data of high-frequency theoretical economies interact with the detrending process? And can the results from this exercise be used to support some detrending methods over others?

The remainder of the paper is as follows. Section 2 introduces the various detrending and TA filters and discusses their cascading. Section 3 investigates the practical ramifications of jointly applying these filters through experiments with known time-series processes and actual macro data. Section 4 applies the analysis to a temporally aggregated, high-frequency RBC model. Section 5 concludes.

2 Temporal Aggregation and Detrending Filters

The most transparent method for investigating the effects of detrending and TA filters is through spectral analysis. A brief review of spectral analysis as it relates to filtering theory is included in Appendix A. For a more thorough coverage of spectral analysis and its applications see Priestley (1988) or Hamilton (1994). In this section, I focus on the properties of the TA and detrending filters individually, as well as their combined or "cascaded" effect.

2.1 Temporal Aggregation Filters

I use the term temporal aggregation to refer to a mapping of either a continuous or short-interval discrete time series process to a longer-interval discrete time series process. Although this paper focuses exclusively on discrete-time processes, the results could be generalized to handle underlying continuous-time processes (e.g., see Sims (1971), Geweke (1978) or Sargent (1987)). Time-aggregated flow data are most often either summed or averaged over time while stock data are typically systematically sampled at regularly spaced discrete-time intervals. Time aggregation, whether a flow or a stock variable, can be thought of as involving a two-step process (Lippi and Reichlin (1991)). In step one, the variable is passed through an aggregation filter. For example, under temporal summing the filter is $h_{TS}(L) = 1 + L + L^2 + ... + L^{n-1}$, where n indicates the number of disaggregate (often referred to as *basic*) periods being aggregated; and under systematic sampling the aggregation filter is $h_{SS}(L) = L^k$, where $k \in \{0, 1, ..., n-1\}$.[1] Step two then involves systematically sampling every n^{th} observation from the $h_{TS}(L)$- or $h_{SS}(L)$-filtered data to form a series of non-overlapping aggregates.

In terms of their spectra, a temporally aggregated series can be related to its disaggregated counterpart using the z and Fourier transforms of $h(L)$ and the folding operator (Sims (1971);

[1] For example, when aggregating a monthly flow variable to an annual frequency, $n = 12$. Aggregating a weekly flow variable to an annual frequency, gives $n = 52$. As $n \to \infty$, one can think of the underlying data generating process as operating in continuous rather than discrete time.

Granger and Siklos (1995)) which is given by

$$F[s_x(\omega)] = \sum_{j=-I}^{I} s_x(\omega + 2\pi j/n) \tag{1}$$

where $F[s_x(\omega)]$ is defined over $\omega \in (-\pi/n, \pi/n]$ and I is the largest number such that $\omega + 2\pi j/n$ falls in the range $(-\pi, \pi]$. The folding operator reflects the well-known *aliasing* identification problem, whereby harmonics at various frequencies cannot be distinguished from one another in sampled data.[2] In essence, aliasing dictates that frequencies outside the $(-\pi/n, \pi/n]$ range of the time-aggregated process get successively folded back into the $(-\pi/n, \pi/n]$ range. The initial folding point, π/n in this context, is sometimes referred to as the Nyquist frequency. Alternatively, the basic and time-aggregated series can be related in the time domain according to their ACGFs. These are respectively

$$g_y(z) = \sum_{j=-\infty}^{\infty} \Gamma_j z^j \text{ and } g_Y(z) = \sum_{j=-\infty}^{\infty} \Gamma_{nj} z^j, \tag{2}$$

where the Γ's are the autocovariances of the $h_{TS}(L)$- or $h_{SS}(L)$-filtered data, and lower- and upper-case letters (i.e., y and Y) refer to the pre-sampled and post-sampled series. From (2), it is clear that time aggregation produces an aggregate ACGF which is simply a sampled version of the basic ACGF at equally spaced n-length intervals.

Returning to the frequency domain, the spectra of a time-aggregated series $\{Y\}$ is then

$$s_Y(\omega) = F[s_y(\omega)] = F[f(\omega)s_x(\omega)], \tag{3}$$

where $f(\omega)$ is the squared gain and $s_Y(\omega)$ is defined over $(-\pi/n, -\pi/n]$. Alternatively, one could redefine $s_Y(\omega)$ in terms of aggregate time and write it as $s_Y(\tilde{\omega})$, where $\tilde{\omega} = \omega n \in (-\pi, \pi]$. Since the folding operator F is an operator rather than a linear filter, there does not exist a unique transfer function for time aggregation of any covariance-stationary time series. Figure 1, however,

[2] When sampling from a continuous time process, I is set to ∞ as frequencies larger than π in magnitude are also aliased (Sims (1971)). Since I am treating the underlying process as one set in discrete time, conceptually, one can imagine that frequencies higher than π have been previously aliased in the sampling from an underlying continuous-time process.

shows the squared TA transfer function associated with $h_{TS}(L)/n$ along with the (shaded) aliased frequencies that will be consecutively folded back (first few folds are shown with vertical lines) onto the range $(0, \pi/n]$. Clearly, the systematic sampling step associated with temporal aggregation produces a loss of information as some higher-frequency harmonics in the basic series will become convoluted with lower-frequency harmonics. The extent to which this convolution is important depends on the nature of the underlying time series, and in particular, on the relative power of the disaggregate spectrum at cycles with periods greater than $2n$ basic periods. The practical importance of this aliasing problem will be explored in more detail below.

These results also extend naturally to a vector time series, with each individual series potentially filtered in a different manner. In this case the cross spectrum between the time aggregates Y_1 and Y_2 is

$$s_{Y_{12}}(\omega) = F[s_{y_{12}}(\omega)] = F[h_1(e^{-i\omega})h_2(e^{i\omega})s_{x_{12}}(\omega)]. \tag{4}$$

2.2 Detrending Filters

As mentioned in the Introduction, there are many different methods that have been applied in business-cycle research to detrend nonstationary time series. In this paper, I focus on three detrending methods – the FD, HP, and BK filters. The properties of these filters are well documented (e.g., see King and Rebelo (1993) and Canova (1998)), so I will only briefly review their properties.[3]

First, the FD filter is given by $h(L) = 1 - L$, which implies that the corresponding squared transfer function is $f(\omega) = 2(1 - \cos(\omega))$. Since $f(0) = 0$, the FD filter removes variation due to the lowest frequency cycles, but unnecessarily magnifies high-frequency variation as can be seen in Figure 2. Another undesirable property of the FD filter is that since it is an asymmetric filter, it introduces a phase shift into the filtered time series.

[3] I do not directly investigate the effects of detrending via regressions of polynomials in time because it is seldom used in modern business-cycle research, in part due to Nelson and Plosser (1982) who find that many macro series are consistent with difference rather than trend-stationary specifications.

Second, the infinite-sample version of the HP filter solves the following problem

$$\min_{\{g_t\}} \left\{ \sum_{t=-\infty}^{\infty} [y_t - g_t]^2 + \lambda \sum_{t=-\infty}^{\infty} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \right\} \tag{5}$$

where λ is an adjustment parameter that governs the growth component g_t by trading off squared deviations from trend against a smoothness constraint. King and Rebelo (1993) show that the filter for the cyclical portion $(y_t - g_t)$ of this filter can be written as

$$h(L) = \frac{\lambda(1-L)^2(1-L^{-1})^2}{1+\lambda(1-L)^2(1-L^{-1})^2}. \tag{6}$$

Given the four first-difference terms in the numerator of (6), the cyclical portion of the HP filter produces a stationary series for any underlying series integrated up to the fourth degree. Furthermore, since (6) is a symmetric filter, it does not introduce any phase shift. When $\lambda = 1600$, the transfer function associated with (6) is an approximate high-pass filter that when applied to quarterly data approximately removes the variation associated with cycles of period longer than 32 quarters. The HP filter for $\lambda = 1600$ is shown in Figure 2.

Third, the BK filter of Baxter and King (1999) is an approximate band-pass filter that is commonly used to eliminate variation associated with cycles of period less than six and more than 32 quarters (i.e., a BP(6,32) filter). The BP(6,32) filter, however, cannot be applied to finite time series because it involves an infinite number of weights in the linear filter.[4] Nevertheless, Baxter and King (1999) show how to calculate an approximate BP(6,32) filter by minimizing the deviation of the ideal and actual weights subject to the constraints that (1) the weights are truncated at K; (2) the filter does not introduce a phase shift; and (3) the associated transfer function removes all variation at the zero frequency (i.e., eliminates certain trends). The BK filter ($K = 6$) that solves this problem is depicted in Figure 2. The BK filter provides a good approximation to the ideal BP(6,32) filter with moderate leakage, compression and exacerbation, using the terminology of Baxter and King (1999). However, the BK filter involves a trade-off. Higher values of K, all else

[4] The HP filter as described above essentially suffers from the same problem since its interpretation as a high-pass filter are based on an infinite sample size. Baxter and King (1999) discuss the consequences of applying the HP filter to finite samples.

equal, lead to better approximations of the ideal BP filter, but it require a loss of $2K$ observations.

2.3 Cascading the Time-Aggregation and Detrending Filters

The previous two subsections discussed the individual effects of temporal aggregation and detrending on a time series. In the context of business-cycle research, however, these two filters are almost always applied together. Despite this fact, there have been very few studies that examine their joint influence. The only papers I am aware of that recognize the dual nature of detrending and time aggregation are Lippi and Reichlin (1991); Ravn and Uhlig (2002); Baxter and King (1999); and Maravall and del Rio (2001). Lippi and Reichlin (1991) are primarily concerned with the sensitivity of trend-cycle variance ratios and measures of persistence to the level of temporal aggregation in the data. The other three papers focus on the relationship between the HP filter and the observational frequency of the data. Baxter and King (1999) briefly mention that it is not clear how to adjust λ across frequencies but suggest that in annual data, $\lambda = 10$ appears to work well, as opposed to the standard values of $\lambda = 100$ or $\lambda = 400$. Ravn and Uhlig (2002) provide a more thorough analytical analysis of how λ should be adjusted across frequencies and suggest, as a rule of thumb, it should be multiplied by n^{-4}. Finally, a working paper by Maravall and del Rio (2001) suggest that the HP filter does not preserve itself under temporal aggregation in the sense that temporally aggregated HP-filtered data cannot be seen as HP-filtered temporally aggregated data. None of these papers, however, provide a systematic analysis of how the most commonly used detrending filters preserve the basic properties of temporally aggregated macro data or whether this criterion can be used to choose amongst the various detrending procedures. To address these issues, one must first examine the joint properties of the detrending and TA filters, which I turn to now.

Typically macro data are aggregated over time before they are detrended. Accordingly, the spectrum of the detrended, time-aggregated series can be related to the spectrum of the basic series according to

$$s_Y(\omega) = f_{DT}(\omega)F[f_{TA}(\omega)s_x(\omega)], \tag{7}$$

where $\omega \in (-\pi/n, -\pi/n]$ and f_{TA} and f_{DT} refer to the squared transfer functions for temporal aggregation and detrending respectively. Expanding (7) using (1) produces

$$s_Y(\omega) = f_{DT}(\omega) f_{TA}(\omega) s_x(\omega) + f_{DT}(\omega) f_{TA}(\omega) \sum_{j=-I, j\neq 0}^{I} s_x(\omega + 2\pi j/n), \tag{8}$$

where again I is the largest number such that $\omega + 2\pi j/n$ falls in the range $(-\pi, \pi]$. As (8) shows, the cascaded transfer function can be decomposed into two parts – the first operates on the lower frequency range of x (i.e., $\omega \in (-\pi/n, \pi/n]$) and the second operates on the higher frequencies that are aliased with $\omega \in (-\pi/n, \pi/n]$.

Figure 3 depicts the cascaded (squared) transfer functions associated with time aggregation ($n = 3$) and the FD, HP and BK filters. As in Figure 1, the aliased frequencies are shaded and will be sequentially folded onto the range $\omega \in (-\pi/n, \pi/n]$. It is clear from Figure 3, which presents the temporal averaging ($h_{TS}(L)/n$) case, that FD- and HP-filtered time-aggregated data will tend to suffer from the aliasing identification problem in that they pass through aliased frequencies. The BK filter, on the other hand, by only passing through cycles with periods between six and 32 quarters, tends to remove the frequencies that suffer the most from the aliasing identification problem in time-aggregated data. A similar story (figure omitted due to space limitations) emerges from the systematic-sampling filter, $h_{SS}(L) = L^k$, where $k \in \{0, 1, ..., n-1\}$. Since the squared transfer function for $h_{SS}(L)$ is always $f_{TA} = 1$, the cascaded transfer functions associated with systematic sampling and the three detrending options essentially become an extended (i.e., $\omega > \pi$) version of Figure 2 with the first folding point at π. In general, the aliasing problem is more severe when detrending systematically sampled as opposed to temporally summed (or averaged) data. It is important to note, however, that irrespective of whether the data are temporally summed or systematically sampled, the BK filter tends to pass through less of the variation associated with aliased frequencies than the HP filter or especially the FD filter.

The analysis above extends to the co-movement between two series in a similar manner. The cross spectrum between detrended, time-aggregated series Y_1 and Y_2 can be related to the cross spectrum of the unfiltered underlying series x_1 and x_2 according to

$$s_{Y_{12}}(\omega) = f_{12}^{DT}(\omega) f_{12}^{TA}(\omega) s_{x_{12}}(\omega) + f_{12}^{DT}(\omega) f_{12}^{TA}(\omega) \sum_{j=-I, j\neq 0}^{I} s_{x_{12}}(\omega + 2\pi j/n), \tag{9}$$

where $f_{12}^{DT}(\omega) = h_{1,DT}(e^{-i\omega}) h_{2,DT}(e^{i\omega})$ and $f_{12}^{TA}(\omega) = h_{1,TA}(e^{-i\omega}) h_{2,TA}(e^{i\omega})$ are the cross transfer functions for detrending and time aggregation respectively. Although variables will typically be detrended in the same fashion, they may be subject to different types of time aggregation. For example, when considering the co-movement between a stock and a flow variable, one series may be systematically sampled while the other is temporally summed. In this case, the time aggregation cross transfer function $f_{12}^{TA}(\omega)$ is likely to involve an imaginary component. Focusing exclusively on the cospectrum (i.e., real part of the cross spectrum) between Y_1 and Y_2, we can write

$$\begin{aligned} c_{Y_{12}}(\omega) \;&=\; \mathrm{Re}[f_{12}^{DT}(\omega) f_{12}^{TA}(\omega)] c_{x_{12}}(\omega) + \mathrm{Re}[f_{12}^{DT}(\omega) f_{12}^{TA}(\omega)] \sum_{j=-I, j\neq 0}^{I} c_{x_{12}}(\omega + 2\pi j/n) - \\ &\left\{ \mathrm{Im}[f_{12}^{DT}(\omega) f_{12}^{TA}(\omega)] q_{x_{12}}(\omega) + \mathrm{Im}[f_{12}^{DT}(\omega) f_{12}^{TA}(\omega)] \sum_{j=-I, j\neq 0}^{I} q_{x_{12}}(\omega + 2\pi j/n) \right\} \end{aligned} \tag{10}$$

where $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ refer to the real and imaginary parts of z, respectively. The basic quadrature, $q_{x_{12}}$, shows up in the formula for the aggregate cospectra, $c_{Y_{12}}$, because the cascaded cross transfer functions may involve imaginary numbers, which will in turn be multiplied by the imaginary number associated with $q_{x_{12}}$. Figure 4 depicts the real and imaginary parts of the transfer function from $s_{x_{12}}$ to $c_{Y_{12}}$ for two detrended, temporally aggregated time series – one temporally averaged using $h_{TS}(L)/n$ and one systematically sampled using $h_{SS}(L) = L^0$, where in both cases $n = 3$. Again, aliased frequencies are shaded and vertical lines represent folding points. What stands out the most in Figure 4 is the fact that the FD filter, and to a lesser extent the HP filter, behave differently when applied to aggregate data than when applied to basic data. For example, even the non-aliased portion of the HP filter is no longer an approximate high-pass filter. Furthermore, both the FD and HP filters suffer more acutely from the aliasing problem than does the BK filter, both in terms of the real and imaginary components of the cross transfer functions.

3 Application to Various Time Series Processes

In this section, I examine a set of known time series processes and a set of actual U.S. time series. Both sets of time series processes are subjected to aggregation and detrending filters to see the practical importance of jointly applying these two filters. The advantage of looking at the first set of data-generating processes is that the population spectra are known at the high-frequency level. The advantage of looking at actual U.S. time series is that they are likely to be more representative of the actual data used by empirical macroeconomists.

3.1 Known Time Series Processes

I begin by considering four simple time series processes to highlight how time aggregation and detrending operate together to effect the spectra of a basic series. The four processes are assumed to take the form

$$x_t = \tau_t + \epsilon_t \tag{11}$$

$$x_t = \tau_t + 0.5x_{t-1} + \epsilon_t \tag{12}$$

$$x_t = \tau_t + \epsilon_t + 0.5\epsilon_{t-1} \tag{13}$$

$$x_t = \tau_t + 0.5x_{t-1} + \epsilon_t + 0.5\epsilon_{t-1} \tag{14}$$

where $\tau_t = a + bt$ is a deterministic linear trend, and ϵ_t is assumed to be mean-zero, white-noise process. Since all four series are non-stationary, they do not have a spectral decomposition. Application of the detrending filters, however, can be thought to work in two steps (Cogley and Nason (1995b)). In step one, the filter makes the series stationary by linearly detrending, and then in step two, the filter acts upon the stationary series that remains. The four processes are aggregated over time using the filter $h_{TS}(L)/n$ with $n = 4$. Conceptually, one can imagine the data as being generated at quarterly intervals but the researcher only observes time-aggregated annual observations. Since the BK filter is designed to only pass through cycles with periods between eight and 1.5 years, it becomes an approximate high-pass filter because cycles with periods of less than two years cannot be identified in annual data. Lastly, the HP adjustment parameter for annual data is set at $\lambda = 6.25$ as suggested by Ravn and Uhlig (2002). Similar annual values for λ are suggested by Baxter and King (1999) ($\lambda = 10$) and Maravall and del Rio (2001) ($\lambda = 7$).

Figure 5 depicts the spectra for detrended, basic $\{x_t\}$ and detrended, time aggregated $\{x_t\}$ using the FD, HP and BK filters. There are several interesting features of the graphs in Figure 5. First, and most starkly, it is clear that the FD filter, when applied to time-aggregated data, greatly distorts the spectral representation of the basic series. Since the FD filter is being applied to time-aggregated rather than basic data, it takes the form $h(L) = 1 - L^n$ with squared transfer function $f(\omega) = 2(1 - \cos(n\omega))$. As a result, the FD filter applied to time-aggregated data experiences a phase shift relative to the basic FD filter that decreases the length of its periods by a factor of n. Whereas the FD filter applied to the basic data tends to monotonically magnify high-frequency variation (see Figure 2), the FD filter applied to time-aggregated data tends to produce spectra with multiple peaks. For the case of $n = 4$, the peaks are associated with cycles that have periodicity of approximately 10 and 30 quarters. Consequently, the combined use of time-aggregated data and first-differencing can produce variation at business-cycle frequencies, even when it is absent in the basic data.[5] Second, there is a moderate amount of aliasing present in the HP-filtered spectra, most of it occurring around the $\omega = \pi/n$ frequency.[6] Third, at least for time-series processes given by (11) - (14), the BK and HP cascaded filters are surprisingly similar and both act as approximate high-pass filters when applied to time-aggregated data.

3.2 Actual U.S. Time Series

Next, I examine how time aggregation and detrending influence the spectra of three U.S. macro time series – GDP, Standard and Poors 500 index (SP500) and consumption. The sources and detailed definitions of these variables are included in Appendix B. GDP is recorded on a quarterly basis while SP500 and consumption are recorded on a monthly basis. For the purpose of this exercise, these are treated as the basic timing intervals or high frequencies. Each series is then artificially aggregated to an annual interval – GDP is temporally summed ($n = 4$), SP500 is systematically

[5] This result is related to the finding of Working (1960), who showed that a first-differenced, time-aggregated random walk contains a moving average term.

[6] To see this most clearly, contrast the HP-TA cascaded (squared) transfer function in Figure 3 with the HP aggregate spectra in Figure 4.

sampled in December ($n = 12$), and consumption is temporally summed ($n = 12$). Before and after aggregation, each series is detrended using the three detrending filters discussed above. Estimated spectra are then produced using a modified Bartlett kernel (see Appendix B for more details) for each of the detrended series at their basic and annual sampling intervals.

Figures 6a, 6b and 6c depict the estimated spectra for GDP, SP500 and consumption for various detrending methods and frequencies. Beginning with the spectra for GDP in Figure 6a, the three panels represent FD-, HP- and BK-filtered data with the quarterly angular frequencies measured along the horizontal axis. The right scale measures the quarterly spectra and the left scale measures the annual spectra. As in Figure 5, the low-frequency spectra are superimposed over the high-frequency spectra, with the vertical line representing the first fold, or stated differently, the shortest cycles (8 quarters $= 2\pi/(\omega = 0.785)$) that are observable in the annual data. The most important feature of Figure 6a is that (despite the fact that GDP is dominated by lower-frequency cyclical variation) the estimated spectra for the time-aggregated annual growth rate of GDP in the first panel clearly distorts the spectral properties of quarterly GDP in terms of their relative magnitudes and timing of peaks and troughs. The same distortion is not present in the HP- and BK-filtered data shown in panels two and three where the quarterly and annual lines have nearly identical shapes.

Figures 6b and 6c differ from Figure 6a in that the two basic series are monthly rather than quarterly so that there is a greater degree of aggregation. In response, I have chosen to show the entire $\omega \in [0, \pi]$ range for only the FD filter, which displays significant high-frequency variation. In the HP and BK filter cases, only a small window $\omega \in [0, \pi/6]$ is shown because there is little-to-no high-frequency variation towards the right-hand-side of the graph. Similar to the GDP case, the FD-filtered time-aggregated annual data experiences substantial distortion relative to the monthly spectra. As expected, the distortion due to the aliasing of high-frequency variation is more acute with the systematically sampled SP500 in Figure 6b than with temporally summed consumption in Figure 6c. The difference between systematic sampling and temporally summing is also highlighted by contrasting the results using the HP and BK filters. In the case of consumption, which is temporally summed, the annual estimated spectra are similar whether one uses the HP or BK filter. However, in the case of systematically sampled SP500, the fact that the HP filter (unlike the

BK filter) is a high-pass filter and the SP500 contains substantial high-frequency volatility, explains why there is noticeable distortion due to aliasing in the annual HP-filtered data but not the annual BK-filtered data. Overall, the findings in Section 3 indicate a potential for serious (moderate) aliasing distortion when using the FD (HP) filter to detrend temporally aggregated data that are associated with significant amounts of underlying high-frequency volatility.

4 Application to Real-Business-Cycle Theory

In this section, I investigate the practical consequences of detrending time-aggregated data within the context of a simple RBC model, similar to the one presented in King, Plosser and Rebelo (1988). The motivation for this exercise is to build a high-frequency theoretical model that might be thought to more closely represent the actual frequency at which economic agents make decisions. Then a calibrated version of the model is solved and used to generate artificial high-frequency data. The high-frequency data are then temporally aggregated (similar to the manner in which the actual data are aggregated), detrended, and compared to the actual U.S. data at the standard quarterly frequency. In this manner, it is possible to examine how closely the detrended, time-aggregated data resemble the underlying properties of the high-frequency data. Similar procedures have been employed by Chari et al. (2000), Aadland (2001), and Cogley and Nason (1995a).

Begin by letting the basic decision interval (assumed to be one week) be indexed by t. A representative agent is assumed to maximize an expected stream of discounted utility

$$U(C, L) = E_t \sum_{s=0}^{\infty} \beta^{t+s} \left[\log(C_{t+s}) + \frac{\phi}{\eta} l_{t+s}^{\eta} \right] \tag{15}$$

by choosing consumption and leisure paths, where E_t is the mathematical expectations operator conditional on all information dated time t and earlier, β a subjective discount factor, C_t consumption, ϕ leisure's weight in total utility, $l_t = (N - L_t)/N$ the proportion of endowed time spent toward leisure, N the endowment of time available for leisure and labor, L_t labor hours, and $1/(1-\eta)$ the intertemporal elasticity of proportional leisure.[7] Consumption is subject to the resource constraint

$$C_t \leq Y_t - I_t, \tag{16}$$

where Y_t is output and I_t gross investment into the capital stock K_t. Capital accumulates according to

$$I_t = K_{t+1} - (1-\delta)K_t, \tag{17}$$

and output is given by the production function

$$Y_t = A_t K_t^{\alpha}(z_t L_t)^{1-\alpha}, \tag{18}$$

where $z_t/z_{t-1} = \exp(\mu)$ is a deterministic labor-augmenting technology process. Total factor productivity (TFP) follows the process $A_t = A_{t-1}^{\rho}\exp(\epsilon_t)$, where ϵ_t is a mean-zero, white noise process with standard deviation σ.

The consumption and labor Euler equations for this problem are

$$C_t^{-1} = \beta E_t\left[C_{t+1}^{-1}(1+\alpha Y_{t+1}/K_{t+1}-\delta)\right] \tag{19}$$

$$(1-\alpha)Y_t/L_t = \phi l_t^{\eta-1}C_t/N. \tag{20}$$

I turn now to calibrating a weekly version of the RBC model. Since weekly data are generally not available, I use an analog version of the model in quarterly time along with some consistency conditions for time aggregation that should be satisfied in the steady state. In other words, I impose that steady-state flow variables in quarterly and weekly time (denoted F and F_*) should obey $F = nF_*$, where $n = 13$. For stocks, the variables should obey $S = S_*$. These constraints are then substituted into the weekly, steady-state version of the RBC model and solved jointly with

[7] The steady-state intertemporal elasticity of labor is also $1/(1-\eta)$ under the assumption that equal proportions of time are spent in leisure and labor activities.

the corresponding quarterly steady-state equations to get

$$\begin{aligned}
\beta_* &= \beta n e^{\mu/n}\left(e^{\mu}+\beta(e^{\mu/n}n-e^{\mu})\right)^{-1} \\
\delta_* &= (\delta/n)+(1-e^{\mu/n})+(1/n)(e^{\mu}-1) \\
\alpha_* &= \alpha \\
\eta_* &= \eta+\log(\phi/\phi_*)/\log(l) \\
\mu_* &= \mu/n,
\end{aligned} \tag{21}$$

where asterisks denote weekly parameters.

There are six unknown parameters in (21) and only five equations. To identify the system I impose an additional restriction that η_* be consistent with microeconomic evidence on high-frequency intertemporal elasticity of substitution in labor. Although the microeconomic evidence does not pin down an exact value for η_*, the evidence confirms the expectation that individuals are more willing to substitute labor across shorter time periods and provides a general guide as to the appropriate magnitude for η_*. I specify that the weekly intertemporal labor supply elasticity is three times as large as its quarterly counterpart, which is roughly in line with microeconomic evidence (e.g., Kimmel and Kniesner (1998), Browning, Hansen and Heckman (1999), Abowd and Card (1989) and MaCurdy (1983)).[8] See Aadland and Huang (2001) for a more detailed discussion of this calibration procedure and the empirical evidence regarding labor supply elasticities across data frequencies. Furthermore, steady-state conditions are not helpful in pinning down weekly values for ρ and σ. As in Chari et al. (2000), I specify that the weekly value of ρ is equal to its quarterly value raised to the 1/13th power. The weekly value of σ is chosen to be consistent with the standard deviation for the error in a quarterly, time-aggregated version of the TFP process, which is set at $\sigma = 0.9$. Further details regarding the calibration of ρ and σ are outlined in Appendix B. Table 1 depicts commonly chosen quarterly parameter values and the implied weekly parameter values.

[8] The qualitative results with respect to temporal aggregation and detrending are robust to reasonable variations in the calibrated value of the weekly labor supply elasticity.

Table 1. RBC Parameters Values

Decision Interval	Parameters β	$1-\delta$	α	η	ϕ	μ	ρ	σ
Quarterly	0.9895	0.975	0.34	0.33	1.11	0.005	0.90	0.90
Weekly	0.9992	0.9981	0.34	0.67	1.39	0.0003	0.9919	0.2618

The linearized version of the model, coupled with the calibrated parameter values in Table 1, are then used to simulate 100 artificial time series. Given that the labor-augmenting technology process (z_t) is trend stationary, series such as output, consumption, investment and real wages will fluctuate about a linear trend. Accordingly, the artificial weekly data are first detrended and then the resulting series are used to estimate standard deviations, correlations, spectra and cospectra for select series. To investigate the effects of time aggregation, the weekly data are first temporally aggregated in a manner consistent with the U.S. data-collection procedures. Then the time-aggregated artificial quarterly data are detrended and used to estimate similar statistics. Lastly, actual detrended quarterly U.S. data are analyzed in a similar fashion. Appendix B provides additional details regarding the US data, detrending procedures, time-aggregation methods, and estimation of the spectra and cospectra.

Begin by considering standard deviations and correlations in the time domain, which are shown in Table 2. The second-moment properties of the simple RBC model are well known (King and Rebelo (1999)). The model does a good job in predicting the fact that consumption is less volatile than output; that investment is more volatile than output; and that consumption, investment and hours worked are procyclical. However, the simple RBC model tends to produce too little volatility in hours worked and predicts grossly counterfactual positive correlations between average labor productivity (or equivalently real wages) and hours worked. To examine the effects of detrending and time aggregation, it is necessary to contrast the results from the weekly and time-aggregated quarterly RBC models. Begin by noticing that the relative volatility predictions are comparable both across decision intervals and detrending methods, although the differences between the weekly and quarterly RBC models are somewhat larger under first-differencing as compared to either the HP or BK filters.

Next consider the cross correlations for the weekly and time-aggregated quarterly RBC models.[9] The HP and BK cross correlations are quite similar across weekly and quarterly models. Taking this together with the standard deviation evidence suggests that the interaction of time aggregation and either the HP or BK filters produces limited distortion of the basic second moments (from the perspective of the time domain). The FD cross correlations, however, are both substantially different than the HP and BK correlations and different across weekly and time-aggregated quarterly RBC data. Whereas the HP- and BK-filtered weekly data display cross correlations that die out gradually over time, the FD-filtered weekly data display strong contemporaneous cross correlations and nearly no lagged (or lead) cross correlations. This is largely due to the fact that the HP and BK filters tend to smooth the cyclical component of the data relative to the FD filter. As for the effects of time aggregation and detrending, which is the primary focus of this paper, the time-aggregated quarterly cross correlations clearly do not retain the cross-correlation patterns present in weekly data. In particular, the combined use of the TA and FD filters tends to give the appearance that there is persistence in the cross correlations over the first couple of leads and lags, where none is present in the weekly data. This can be seen even more clearly in Figure 7, which depicts the 32 weekly lead and lag cross correlations for various series (as opposed to only the five sampled correlations shown in Table 2). The sharp peak in the FD cross correlations and the relatively smooth decay of the HP and BK cross correlations, when compared to the quarterly cross correlations in Table 2, confirm that time aggregation tends to distort the cross correlation pattern in FD-filtered data but not in HP- or BK-filtered data.

Figure 8 depicts detrended weekly and quarterly estimated spectra for output in the RBC and US economies. The most remarkable feature in Figure 8 is the difference between first-differenced weekly and quarterly RBC spectra. The spectra for weekly output in the RBC economy is nearly a "flat line", implying that detrended weekly RBC output is approximately white noise. The fact that standard RBC models produce little persistence in output is a well-known result (Cogley and Nason (1995a)). The surprising result is that detrended quarterly (i.e., weekly summed) RBC output displays variation at business-cycle frequencies. This is similar in spirit to the findings of

[9] To make the weekly cross correlations comparable to the quarterly correlations, I sample every 13th weekly cross correlation.

Cogley and Nason (1995b) regarding the HP filter, except for the fact that the results presented here do not rely on difference-stationary specifications. The oscillatory and gradual downward trend of the quarterly RBC spectra over $\omega \in (0, \pi/13]$ are the result of time aggregation (which tends to magnify low-frequency variation), first-differencing in quarterly time (i.e., applying $h(L) = 1 - L^{13}$ which produces the oscillations), and the aliasing of higher-frequency variation that occurs due to systematic sampling.

The other notable feature of Figure 8 is the surprising similarity in the shapes of the HP- and BK-filtered spectra for weekly and quarterly output, both across sampling intervals and across filters. These similarities are driven by two forces. First, the spectral shape for filtered RBC output reflects the fact that it is both nearly difference stationary (i.e., has an autoregressive root very near one) and the model fails to propagate the technology shocks through time. This implies that the spectra for RBC output primarily reflect the shape of the asymmetric $S(L)$ filter that remains after first-differencing (see Cogley and Nason (1995b), page 259). Second, the similarity of the HP- and BK- filtered output is due to the fact that output does not display much high-frequency variation and therefore the high-pass HP filter and band-pass BK filter behave similarly. As Baxter and King (1999) mention, this is not true for series such as inflation (and SP500 from Section 3.2) which display substantially more high-frequency variation. Therefore, according to the criteria that detrending filters do not distort basic spectra when applied to time-aggregated data, these results provide strong evidence in favor of using either the HP or BK filters over first-differencing.

The conclusions are much the same for the cospectra. Figure 9 shows the cospectra between detrended average labor productivity and hours worked in the RBC and US economies. First notice that the primary peak in the cospectra for the US data is below zero, which reflects the negative correlation between the two series, while the peaks in the RBC cospectra are above zero, reflecting the strong positive correlations predicted by the model. This mismatch between theory and observation persists across detrending methods and time aggregation. More importantly, however, notice that the shapes of the weekly and quarterly RBC cospectra are much closer for HP- and BK-filtered data as opposed to first-differenced data. Once again, this provides support for the use of either the HP or BK filters rather than the FD filter when detrending time-aggregated data.

5 Conclusion

This paper attempts to shed light on an important practical problem for business-cycle researchers: "Is time aggregation an important ingredient in the decision of how to detrend macro data?" The results indicate it is. In particular, although the FD filter is easy to apply across frequencies, it tends to produce the greatest amount of distortion to the basic spectral properties of a time-aggregated series. This occurs because the FD filter tends to magnify the same high-frequency variation that is lost (or more accurately, aliased) in the aggregation process. As a result, it is possible that a researcher may discover substantial variation in economic series at business-cycle frequencies that are exclusively the by-product of time aggregation and detrending. This is similar to the results of Cogley and Nason (1995b), who find spurious cycles in difference-stationary data that have been detrended by means of the HP filter. The HP filter, on the other hand, provides limited distortion of the basic series and moderate amounts of aliasing of higher-frequency variation. However, it is not immediately clear how the value of the smoothing parameter should change across sampling intervals. The accepted value for λ in quarterly data is 1600, but there is no widely accepted value for λ in, say, monthly or annual data, although recent research by Ravn and Uhlig (2002) suggest a reasonable rule of thumb. Overall, the BK filter appears to be the least distortionary and easiest to adjust across frequencies. Of course, there are many other criteria for choosing a detrending method such as what aspect of business-cycle fluctuations are important (e.g., labor hoarding, monetary policy, etc.) or how robust the detrending method is to the nature of the nonstationarity. That said, given the pervasive use of temporally aggregated and systematically sampled data in macroeconomics, the results from this paper argue for caution when using detrending filters that either exacerbate (e.g., FD filter) or simply pass-through (e.g., HP filter) high-frequency variation.

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Appendices

A Review of Spectral Analysis

Assume $\{x_t\}_{t=-\infty}^{\infty}$ is a covariance-stationary time series with absolutely summable autocovariances given by $\gamma_j = E[(x_t - \mu_x)(L^j x_t - \mu_x)]$, where $L^j x_t = x_{t-j}$, $\mu_x = E[x_t]$, and $E[\cdot]$ is the mathematical expectation operator. Cramér's spectral representation theorem (Priestley (1988)) guarantees that x_t can be decomposed into a weighted sum of sine and cosine waves according to

$$x_t = \mu_x + \int_{-\pi}^{\pi} [\alpha_1(\omega)\sin(\omega t) + \alpha_2(\omega)\cos(\omega t)]d\omega, \tag{22}$$

where the weights $\alpha_1(\omega)$ and $\alpha_2(\omega)$ are serially and mutually uncorrelated random variables over the angular frequencies ω measured in radians. Using the autocovariance generating function (ACGF) for x_t, $g_x(z) = \sum_{j=-\infty}^{\infty} \gamma_j z^j$, De Moivre's theorem and some simple trigonometry, one can generate the population spectrum

$$s_x(\omega) = (2\pi)^{-1}\{\gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \cos(\omega j)\}. \tag{23}$$

For covariance-stationary time series, $s_x(\omega)$ will be symmetric about $\omega = 0$ with a period equal to 2π. As a result, all relevant information regarding the spectrum is contained within frequencies between 0 and π. Furthermore, we can recover the k^{th} autocovariance for x using $s_x(\omega)$ according to

$$\gamma_k = \int_{-\pi}^{\pi} s_x(\omega)\cos(\omega k)d\omega. \tag{24}$$

Equation (24) indicates the practical use of the spectrum. For example, we are able to decompose the fraction of the variance in x (γ_0) due to cycles with frequencies between ω_1 and ω_2 by integrating the area under the spectrum between ω_1 and ω_2 (i.e., $\int_{\omega_1}^{\omega_2} s_x(\omega)d\omega$). In particular, this allows us to isolate the amount of variation in a stationary series that is due to cycles at business-cycle frequencies.

Time-invariant linear filters can be introduced by pre-multiplying x_t by a polynomial in the lag operator L, $h(L)=\sum_{j=-\infty}^{\infty}h_jL^j$ with h_j absolutely summable. This generates

$$y_t = h(L)x_t, \tag{25}$$

where y_t is the filtered version of x_t. Using (25) and $g_x(z)$ it can be shown that a linear filter has the effect of multiplying the spectrum for x by $f(\omega)=h(e^{-i\omega})h(e^{i\omega})$, which is often referred to as the squared gain or squared modulus of the transfer function. Therefore, the spectrum for y is given by $s_y(\omega)=f(\omega)s_x(\omega)$. If $f(\omega)>1$, then the filter magnifies the variation in x at frequency ω and if $f(\omega)<1$, then the filter diminishes variation in x at frequency ω.

Linear filters can also be expressed in their polar form. Writing $h(\omega)$ in its polar form produces

$$h(\omega)=R\exp(-i\varphi(\omega)), \tag{26}$$

where $R=\sqrt{f(\omega)}$ represents the gain of the filter and $\varphi(\omega)$ represents the phase shift introduced by the filter $h(L)$. The HP filter, for example, is a symmetric filter and does not introduce a phase shift (i.e., $\varphi(\omega)=0$, for all ω) while temporal summing and systematic sampling are typically asymmetric filters, so they are likely to introduce a phase shift.

Business-cycle researchers are also interested in the co-movement between variables. In order to examine the impact of filtering on the co-movement between two variables, let $h(L)$ be a 2×2 matrix filter

$$\begin{bmatrix} h_1(L) & 0 \\ 0 & h_2(L) \end{bmatrix} \tag{27}$$

that operates on the vector time series $x_t=(x_{1,t},x_{2,t})'$. Then the (2×2) matrix spectrum for $y_t=(y_{1,t},y_{2,t})'$ can be written as $s_y(\omega)=h(e^{-i\omega})s_x(\omega)h(e^{i\omega})'$. The main-diagonal elements are the spectra for y_1 and y_2, while the off-diagonal elements are the cross-spectrum between y_1 and y_2. For example, the cross-spectrum between y_1 and y_2 ($s_{y_{12}}(\omega)$) can be written in terms of the cross-spectrum between x_1 and x_2 as $s_{y_{12}}(\omega)=h_1(e^{-i\omega})h_2(e^{i\omega})s_{x_{12}}(\omega)$. As expected, both $h_1(L)$ and $h_2(L)$ are important in determining the nature of the co-movement between the filtered variables y_1 and y_2. The cross spectra $s_{y_{12}}(\omega)$ and $s_{x_{12}}(\omega)$ are generally complex numbers and can be decomposed as

$$s_{x_{12}}(\omega) = c_{x_{12}}(\omega) + iq_{x_{12}}(\omega) \tag{28}$$

where $c_{x_{12}}(\omega)$ is referred to as the cospectrum and $q_{x_{12}}(\omega)$ as the quadrature. Since $q_{x_{12}}(\omega)$ integrates to zero over $(-\pi, \pi]$, the contemporaneous covariance between x_1 and x_2 is given by $\int_{-\pi}^{\pi} c_{x_{12}}(\omega)d\omega$.

B Technical Details of Detrending and Aggregation

This appendix describes the technical details for detrending and time aggregation using either U.S. or RBC data. The intent is to be sufficiently thorough that an individual could readily replicate the results of Sections 3 and 4.

B.1 U.S. Data Definitions and Sources

The data are taken from FRED, the Federal Reserve Economic Database of the St. Louis Federal Reserve Bank and span the first quarter of 1950 through the fourth quarter of 1999. The series selected are as follows, with FRED mnemonic in parentheses: output is measured as real gross domestic product (GDPC1); consumption is measured as real personal consumption expenditures of all goods and services (PCECC96) less consumption expenditures on durable goods (PCEDC96); investment is measured as the sum of real fixed private investment (FPIC1), real public gross investment (DGIC96 + NDGIC96 + SLINVC96), and real personal consumption expenditures on durable goods (PCDGCC96); total hours are measured as the product of average weekly hours in private nonagricultural establishments (AWHNOG) and total nonfarm payroll employees (PAYEMS); stock prices are measured by the total returns on Standard and Poors 500 composite (TRSP500); and lastly, average productivity is calculated by dividing output by total hours. All the variables have been seasonally adjusted at the source; output, private consumption, investment and government consumption are measured in chain-weighted 1996 dollars; and all but average productivity have been transformed into per-capita terms using the entire civilian, non-institutionalized population that is sixteen and over (CNP16OV).

B.2 Calibrating Weekly ρ and σ

Begin by assuming that total factor productivity (denoted in logarithms) follows a first-order autoregressive process in weekly time

$$A_t = \rho_* A_{t-1} + \epsilon_t, \tag{29}$$

where ϵ_t is a mean-zero, white-noise process with standard deviation σ_*. Repeated substitutions for A on the right-hand-side of (29) produces

$$A_t = \rho_*^{13} A_{t-13} + \sum\nolimits_{s=0}^{12} \rho_*^s \epsilon_{t-s} = \rho A_{t-13} + \nu_t. \tag{30}$$

Treating (30) as the quarterly process for total factor productivity, then we have a straightforward mapping from the quarterly parameters to the weekly parameters. Given a value for ρ, then the weekly autoregressive coefficient is given by $\rho_* = \rho^{1/13}$. Furthermore, given a measure of the standard deviation of ν_t, σ, then the weekly standard deviation is given by

$$\sigma_* = \sqrt{\sigma^2 \frac{(1-(\rho_*^2)^{13})}{(1-\rho_*^2)}}.$$

B.3 Time-Aggregation Procedures

It is important that the time aggregation procedures for the RBC data match the U.S. sampling and aggregation procedures. Series taken from the National Income and Product Accounts (NIPA)–output, consumption, and investment–are collected using a wide range of sources and at varying points in the quarter; see the BEA's website (http://www.bea.doc.gov/bea/mp.htm) for more details. The artificial data for these series were aggregated using the temporal aggregation operator, $h_{TS}(L) = 1 + L^1 + ... + L^{12}$. Series taken from the Bureau of Labor Statistics (BLS)–average weekly hours, employment, and the population–are collected from the household and establishment surveys. The following excerpt is taken from the BLS website (http://stats.bls.gov:80/) regarding their sampling methods:

> For both surveys, the data for a given month relate to a particular week or pay period. In the household survey, the reference week is generally the calendar week that

contains the 12th day of the month. In the establishment survey, the reference period is the pay period including the 12th, which may or may not correspond directly to the calendar week.

In accordance with BLS techniques, average quarterly hours worked and population data generated from the model were thus treated as being systematically sampled from the second week of the month and then averaged over the three months in the quarter, that is, $h_{SS}(L) = (L^3 + L^7 + L^{11})/3$.

B.4 Empirical Detrending Procedures

Before detrending, all U.S. variables were transformed into natural logarithms. The variables within the RBC model are measured as log deviations from their steady state. Therefore, the artificial RBC variables were made comparable to their U.S. counterparts by adding the logged steady-states to the proportional deviations from the steady-state values. While the FD filters for weekly and quarterly data are straightforward to apply (i.e., $h(L) = 1 - L$ and $h(L) = 1 - L^{13}$ respectively), the ideal properties of the HP and BK filters require an infinitely large sample. To apply the HP filter in finite samples, I use the Regression Analysis of Time Series (RATS) procedure *hpfilter.src* with $\lambda = 1600 \times 4^{-4}$ for annual data, $\lambda = 1600$ for quarterly data, and $\lambda = 1600 \times (1/13)^{-4}$ for weekly data, as suggested by Ravn and Uhlig (2002). For the BK filter, I apply the RATS procedure *bkfilter.src* available at http://www.estima.com/procindx.htm. For quarterly data, the BK filter parameters are set at upper = 6, lower = 32, arpad = 8, and nma = 24. For annual and weekly data, the parameters are set at 0.25 and 13 times their quarterly values, respectively.

B.5 Estimation of Spectra and Cross Spectra

The spectra for the U.S. and RBC artificial data are estimated using the modified Bartlett kernel (Hamilton (1994), pp. 330-332):

$$s_x(\omega) = \frac{1}{2\pi}\left(\hat{\gamma}_0 + 2\sum_{j=1}^{h}(1 - \frac{j}{h+1})\hat{\gamma}_j \cos(j\omega)\right),$$

where $h = 24 \times 0.25 = 6$ for annual data, $h = 24$ for quarterly data and $h = 24 \times 13 = 312$ for weekly data and $\hat{\gamma}_s$ indicates the s-lag autocorrelation coefficient for x_t. The cross spectra are estimated similarly using

$$s_{xy}(\omega) = \frac{1}{2\pi}\left(\sum_{j=-h}^{h}(1 - \frac{j}{h+1})\hat{\gamma}_{xy}(j)\cos(j\omega)\right),$$

where $\hat{\gamma}_{xy}(s)$ indicates the correlation between x_t and y_{t+s}.

Table 2. Select Second-Moment Statistics for the US and RBC Economies

Standard Deviations Relative to y	US Data			Weekly RBC Model			Time Aggregated Quarterly RBC Model		
	FD Filter	HP Filter	BK Filter	FD Filter	HP Filter	BK Filter	FD Filter	HP Filter	BK Filter
std(c)	0.662	0.605	0.581	0.340	0.404	0.422	0.385	0.410	0.424
std(dk)	2.366	2.359	2.238	2.973	2.894	2.867	2.968	2.894	2.865
std(n)	1.050	1.096	1.125	0.496	0.481	0.475	0.464	0.474	0.473
std(y/n)	0.727	0.548	0.523	0.505	0.543	0.555	0.531	0.547	0.557

Correlations	Lags	FD Filter	HP Filter	BK Filter	FD Filter*	HP Filter*	BK Filter*	FD Filter	HP Filter	BK Filter
	-2	0.257	0.648	0.676	-0.002	0.297	0.444	-0.038	0.329	0.450
	-1	0.388	0.783	0.831	-0.003	0.580	0.738	0.198	0.642	0.739
corr(c,y)	0	0.585	0.815	0.859	0.996	0.933	0.921	0.943	0.928	0.919
	+1	0.271	0.642	0.707	0.007	0.760	0.929	0.347	0.833	0.929
	+2	0.091	0.455	0.469	0.007	0.591	0.786	0.111	0.648	0.791
	-2	0.217	0.593	0.659	-0.001	0.479	0.694	-0.018	0.532	0.699
	-1	0.339	0.759	0.824	-0.001	0.713	0.923	0.234	0.794	0.923
corr(dk,y)	0	0.724	0.851	0.865	0.999	0.989	0.985	0.978	0.984	0.984
	+1	0.328	0.701	0.738	-0.004	0.639	0.840	0.178	0.716	0.841
	+2	0.089	0.507	0.511	-0.004	0.357	0.544	-0.075	0.400	0.550
	-2	0.109	0.365	0.445	-0.001	0.496	0.714	-0.018	0.510	0.659
	-1	0.432	0.656	0.725	-0.001	0.717	0.925	0.138	0.751	0.891
corr(n,y)	0	0.750	0.868	0.885	0.999	0.974	0.966	0.914	0.967	0.972
	+1	0.482	0.811	0.842	-0.006	0.605	0.799	0.400	0.772	0.852
	+2	0.259	0.653	0.672	-0.005	0.311	0.487	-0.099	0.425	0.571
	-2	0.019	0.051	-0.072	-0.007	0.208	0.346	-0.136	0.310	0.431
	-1	-0.074	-0.185	-0.295	-0.009	0.515	0.675	0.348	0.674	0.735
corr(y/n,n)	0	-0.407	-0.423	-0.457	0.996	0.909	0.883	0.885	0.911	0.901
	+1	-0.143	-0.467	-0.525	0.001	0.721	0.903	0.179	0.764	0.882
	+2	-0.188	-0.474	-0.510	0.002	0.548	0.755	0.029	0.574	0.712

Notes: The variables c, dk, n, and y/n refer to consumption, investment, labor hours and average labor productivity, respectively. Std(x) refers to the standard deviation of detrended x relative to the standard deviation in detrended y. Corr(x,z) refers to the cross correlation between detrended x and detrended z. Lag j refers to the correlation between contemporaneous x and z lagged j periods.

*The reported cross correlations refer to every 13^{th} weekly lagged cross correlation. That is, the cross correlations for lags=-2,-1,0,+1,+2 correspond to weekly lags=-26,-13,0,+13,+26.

Figure 1. Temporal Aggregation Transfer Functions

Aliased frequencies are shaded, Vertical lines denote folding points

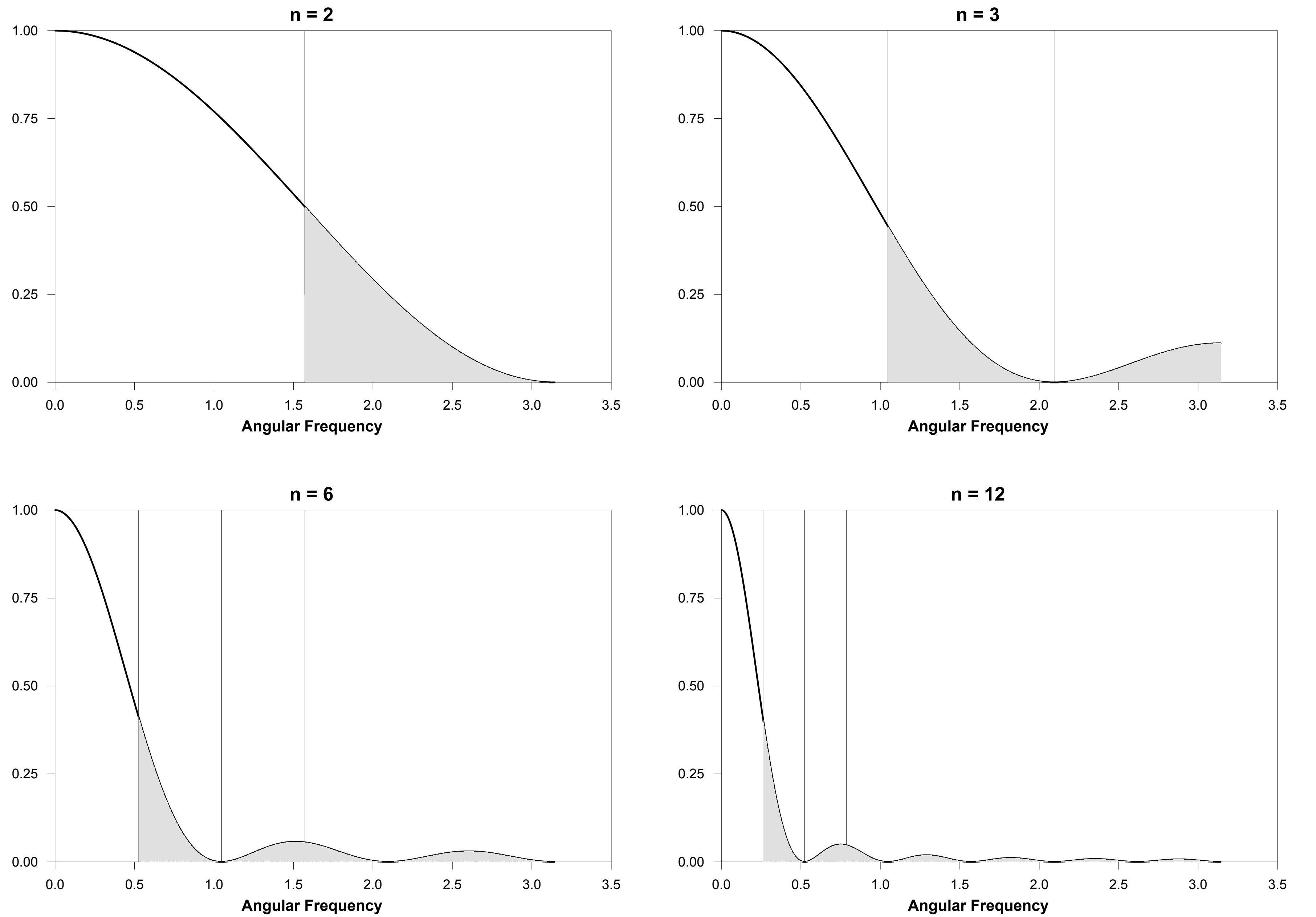

Figure 2. Detrending Transfer Functions

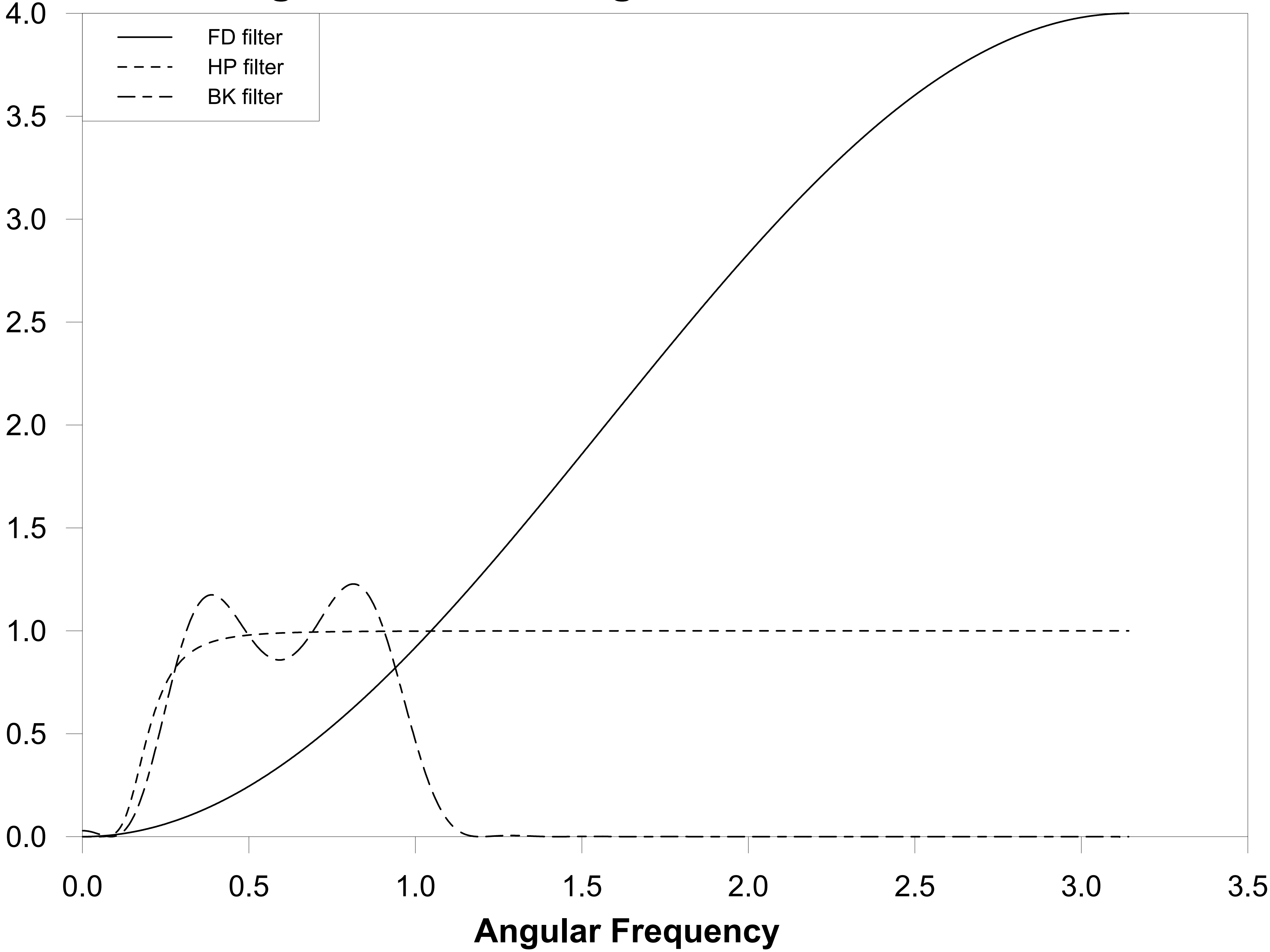

Figure 3. Cascaded Transfer Functions (n = 3)

Aliased frequencies are shaded, Vertical lines denote fold points

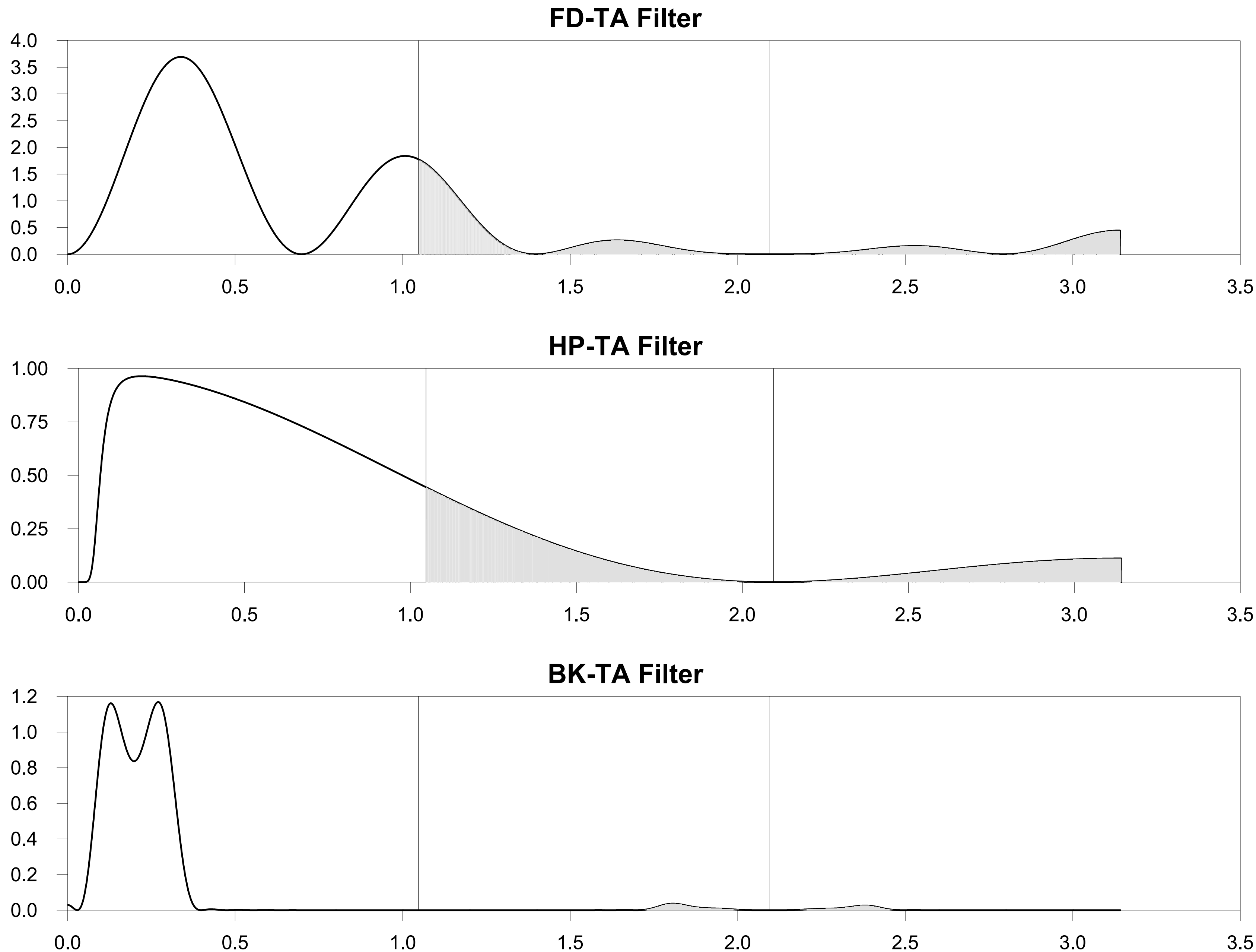

Figure 4. Cascaded TA-SS Cospectra Transfer Functions (n = 3)

Aliased frequencies are shaded, Vertical lines denote fold points

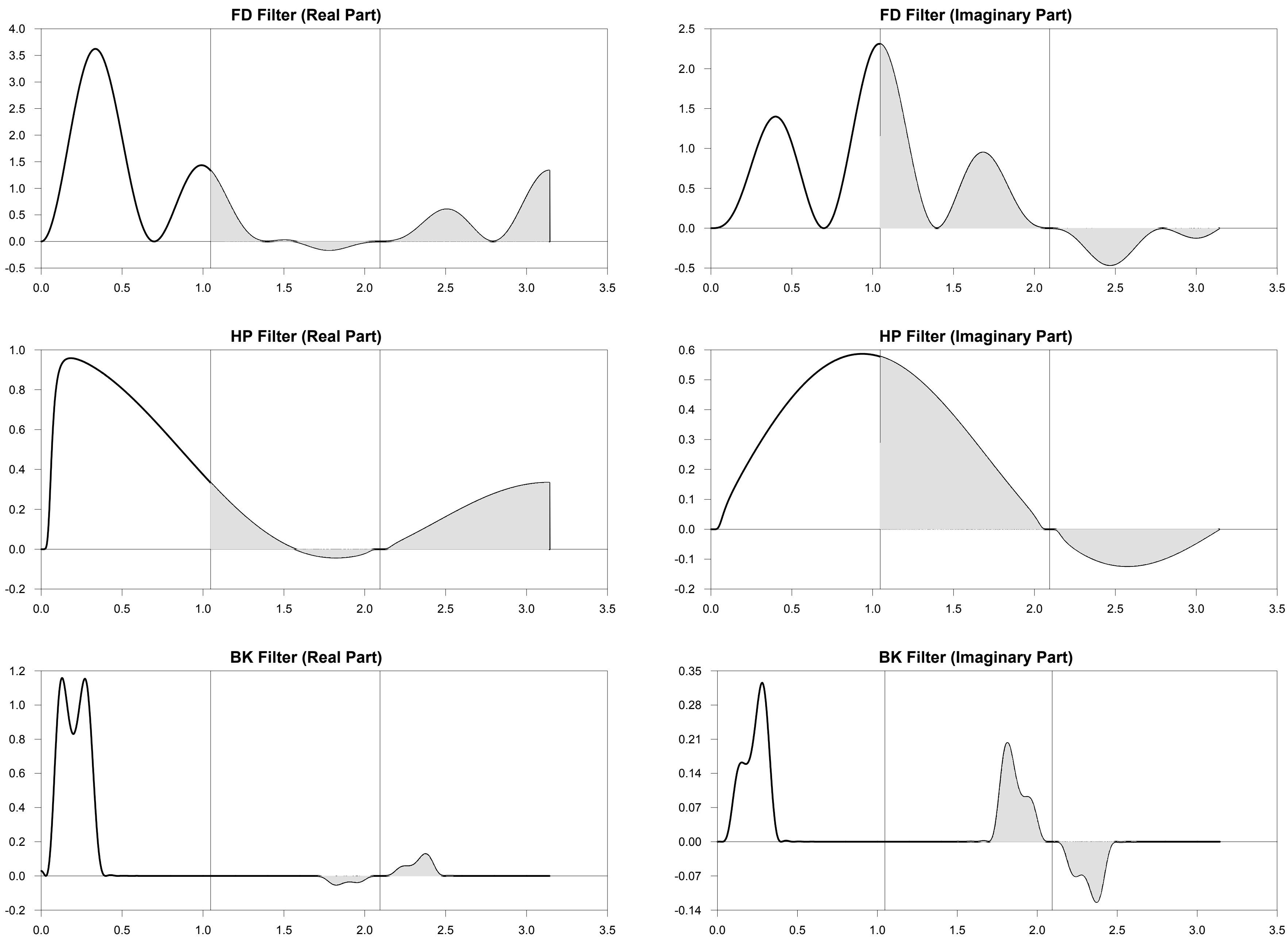

Figure 5. Detrended Basic and Time Aggregated Spectra (n = 4)

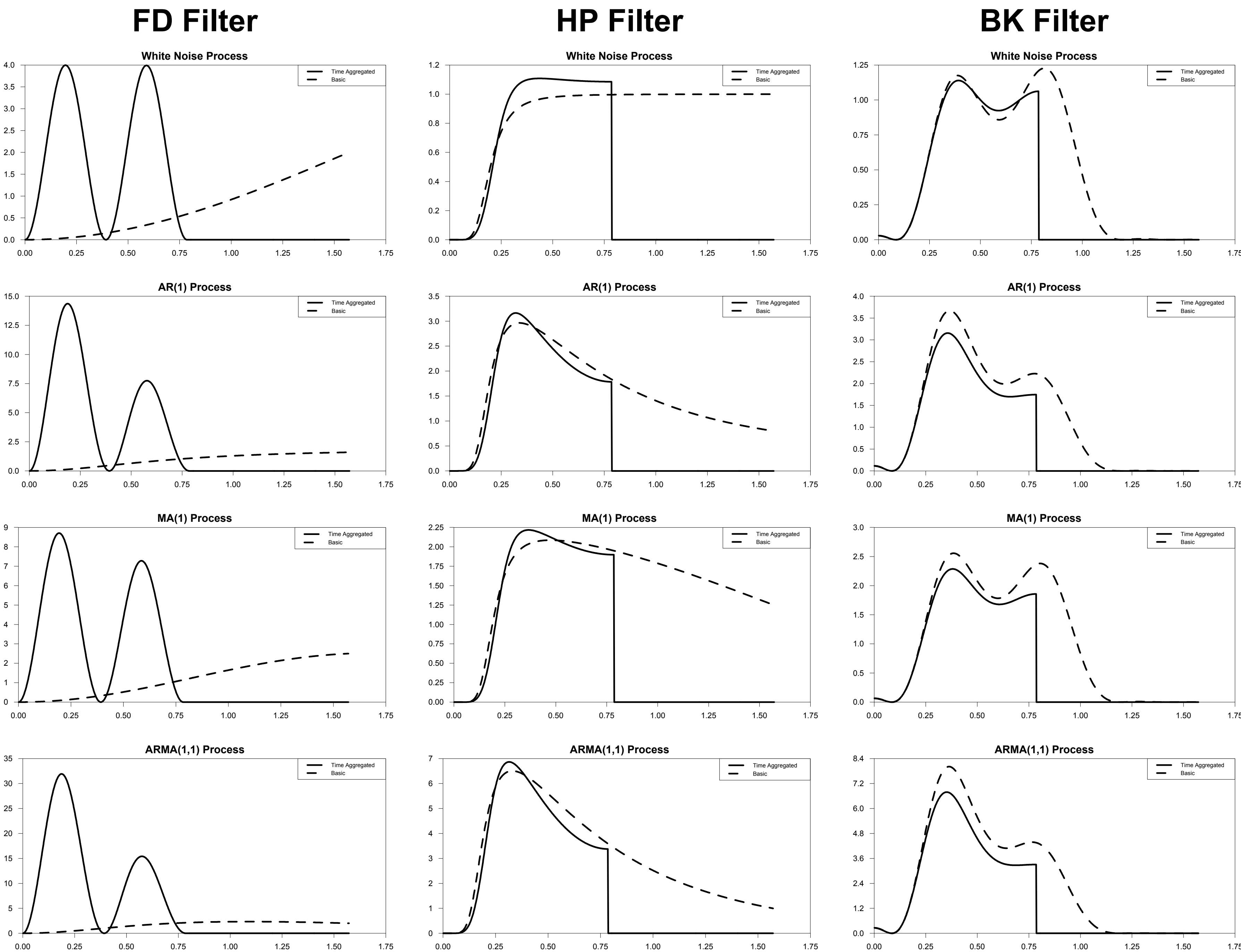

Figure 6a. Spectra for Detrended GDP

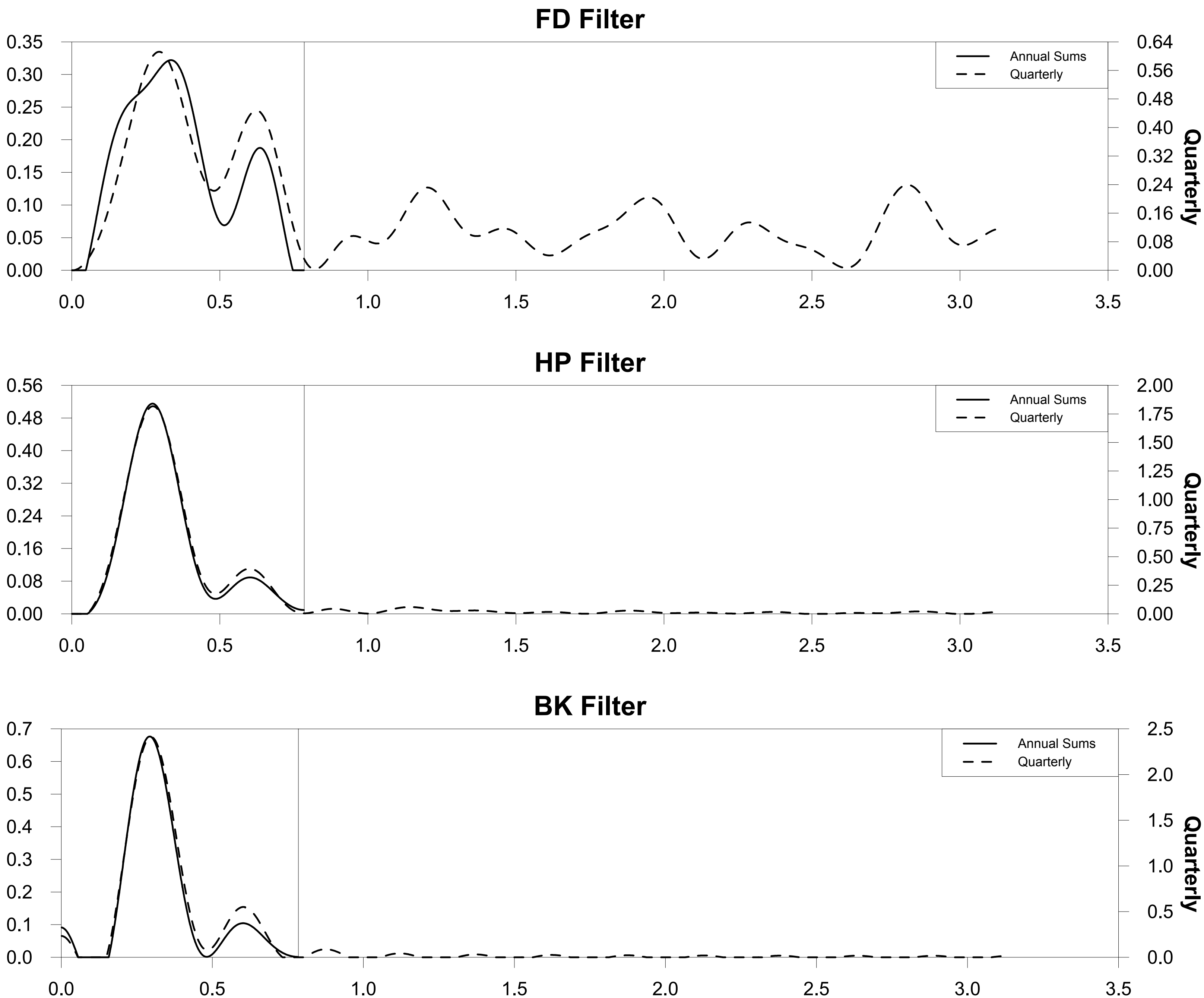

Figure 6b. Spectra for Detrended S&P 500

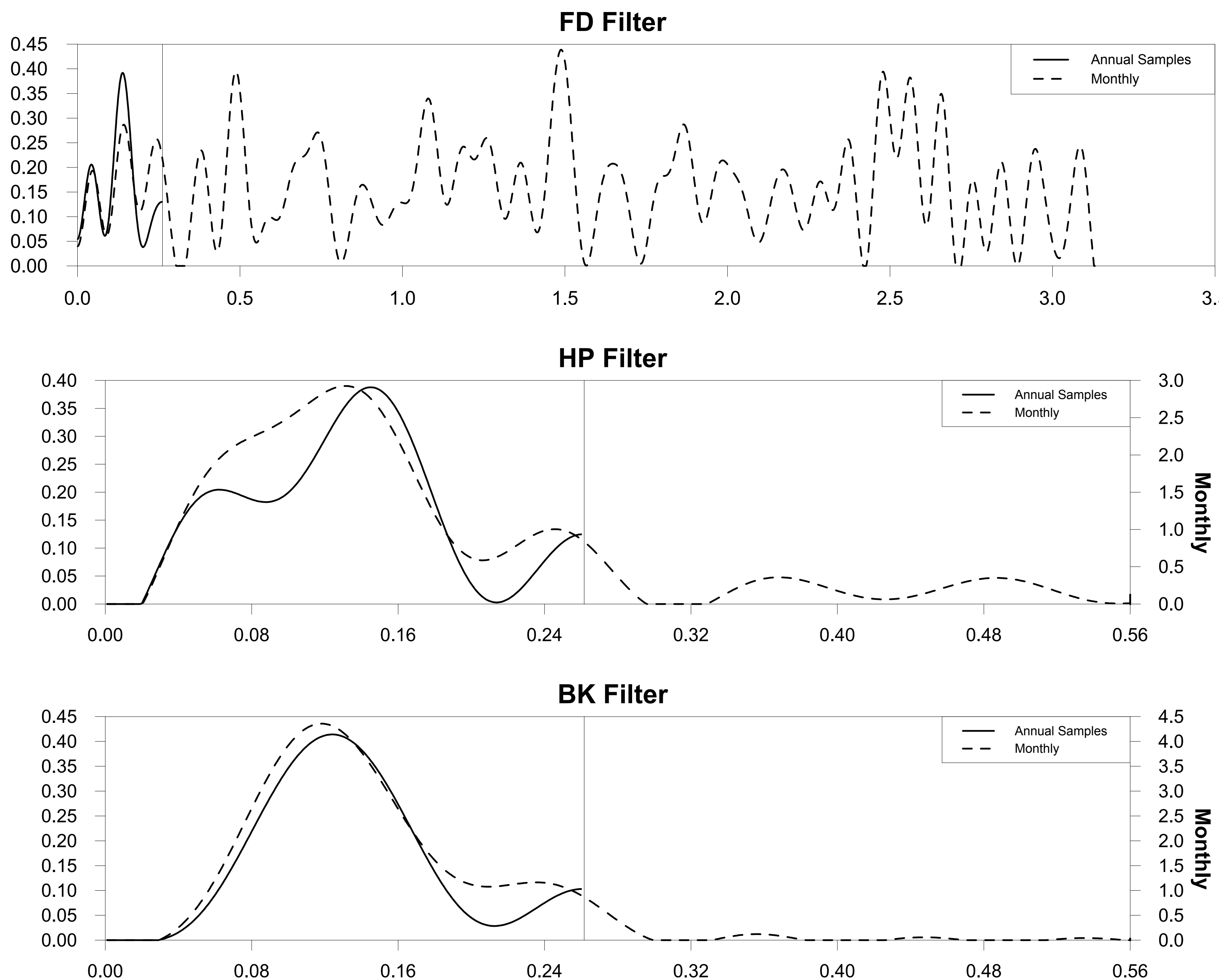

Figure 6c. Spectra for Detrended Consumption

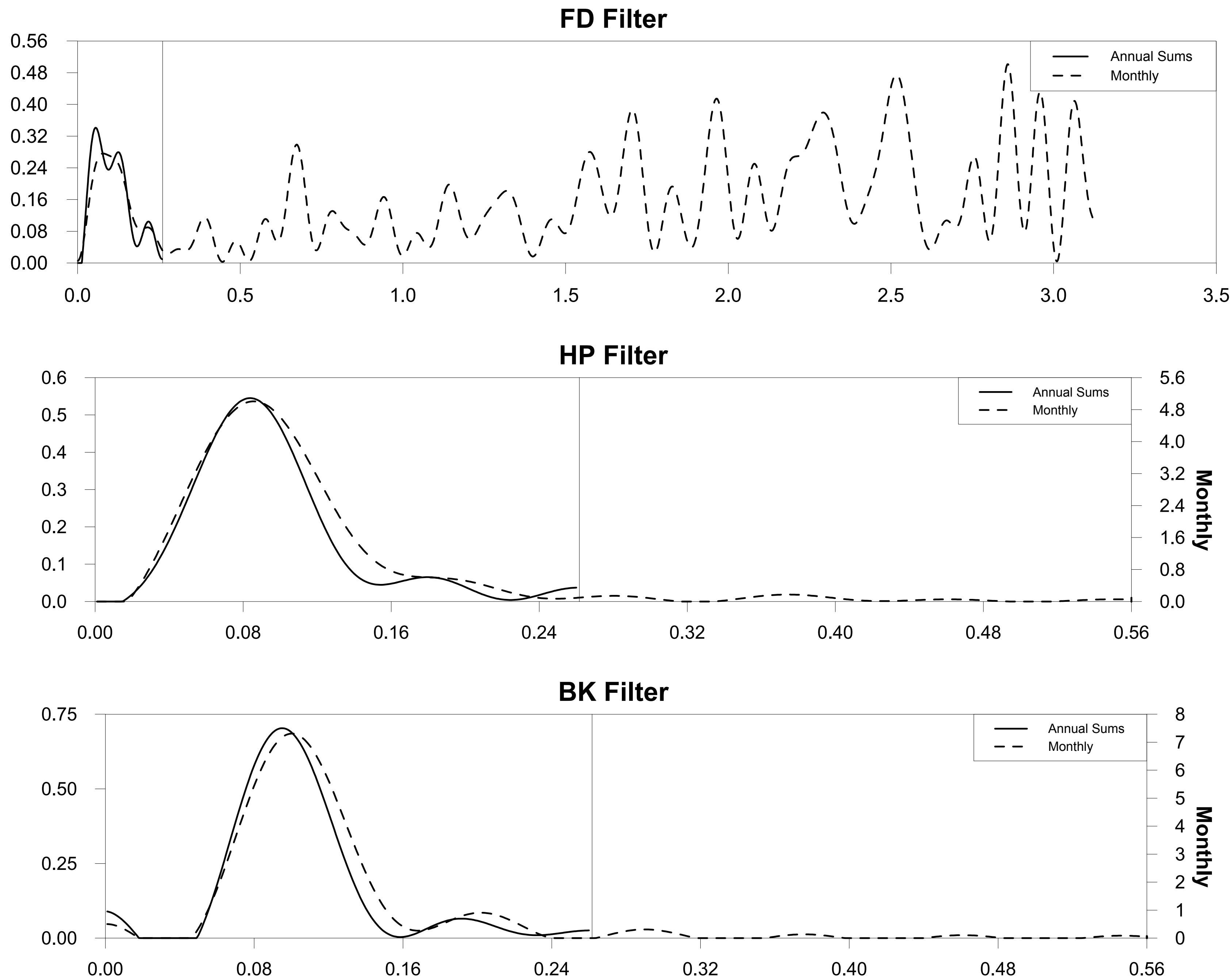

Figure 7. Weekly Lagged Cross Correlations for RBC Model

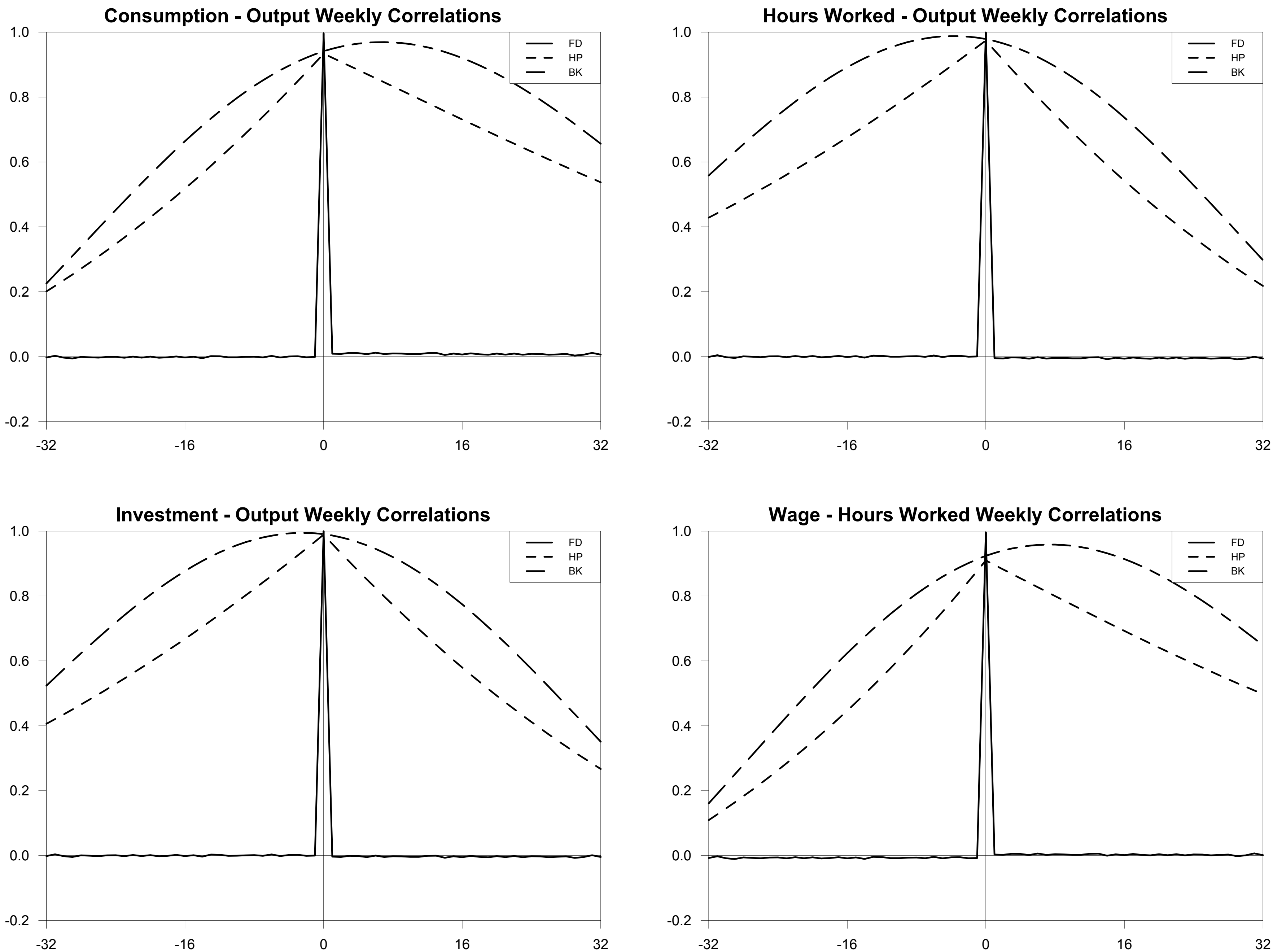

Figure 8. Spectra for Detrended RBC and US Output

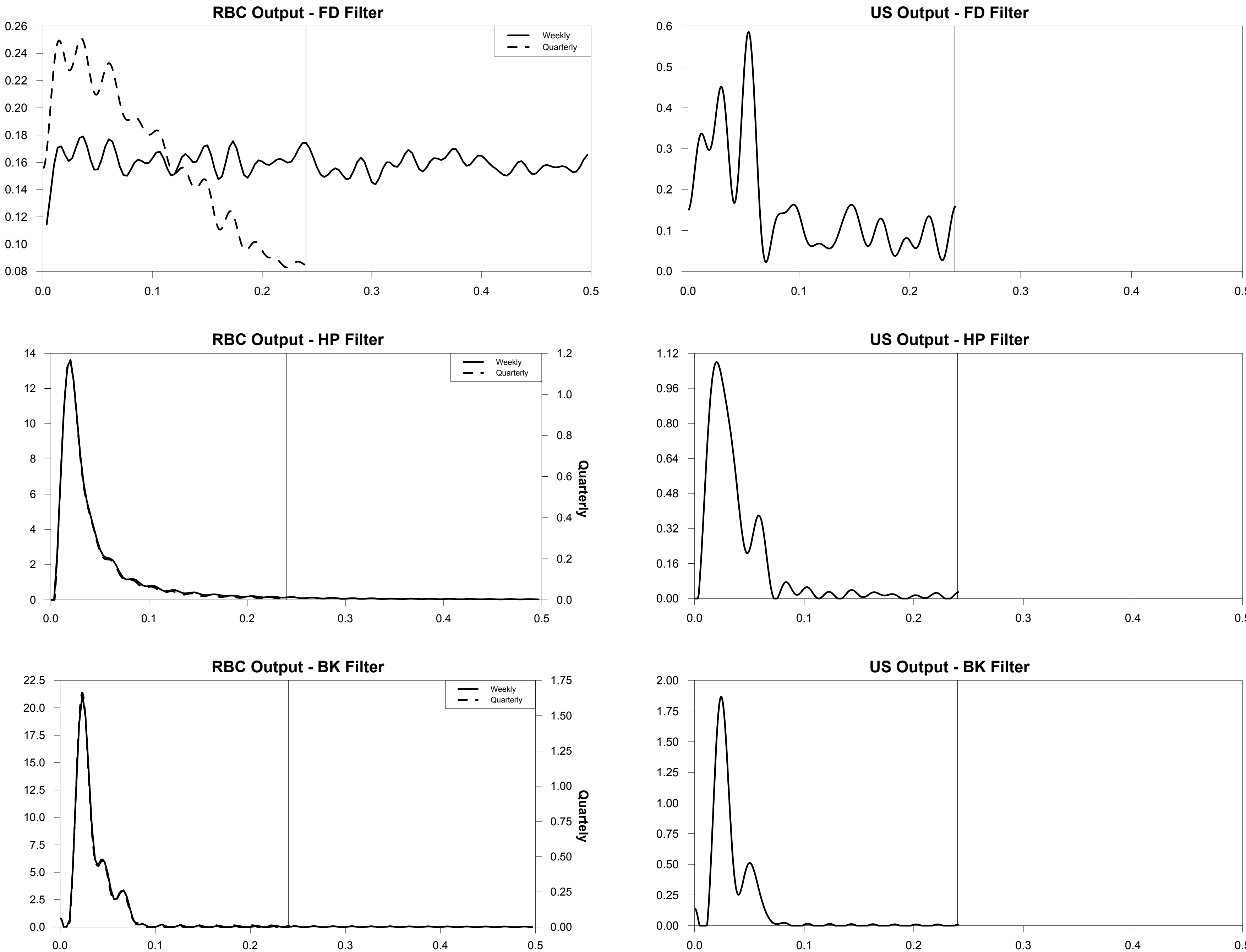

Figure 9. Cospectra for Detrended Wages and Hours Worked